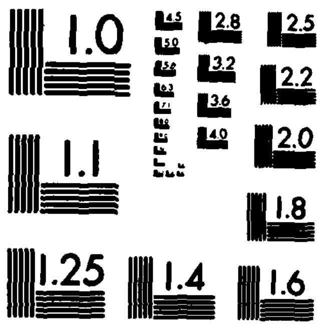
AD-A146 837 AN OPERATOR METHOD FOR COMPUTING THE ASYMPTOTICS OF A COLLISION RESOLUTIO..(U) MASSACHUSETTS UNIV AMHERST DEPT OF MATHEMATICS AND STATISTICS.. W A ROSENKRANTZ 1/1

UNCLASSIFIED 29 AUG 84 AFOSR-TR-84-0919 AFOSR-82-0167 F/G 12/1 NL

END


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

# An Operator Method for Computing the Asymptotics of a Collision Resolution Interval

Walter A. Rosenkrantz(1)

Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA 01003



(1) Research supported by AFOSR Grant #82-0167.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

# REPORT DOCUMENTATION PAGE

| | |
|---|---|
| 1a. REPORT SECURITY CLASSIFICATION<br>UNCLASSIFIED | 1b. RESTRICTIVE MARKINGS |
| 2a. SECURITY CLASSIFICATION AUTHORITY | 3. DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release; distribution unlimited |
| 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4. PERFORMING ORGANIZATION REPORT NUMBER(S) | 5. MONITORING ORGANIZATION REPORT NUMBER(S)<br>**AFOSR-TR- 84-0919** |
| 6a. NAME OF PERFORMING ORGANIZATION<br>University of Massachusetts<br>6b. OFFICE SYMBOL (If applicable) | 7a. NAME OF MONITORING ORGANIZATION<br>Air Force Office of Scientific Research |
| 6c. ADDRESS (City, State and ZIP Code)<br>Department of Mathematics and Statistics<br>Amherst MA 01003 | 7b. ADDRESS (City, State and ZIP Code)<br>Directorate of Mathematical and Information Sciences, Bolling AFB DC 20332 |
| 8a. NAME OF FUNDING/SPONSORING ORGANIZATION<br>AFOSR<br>8b. OFFICE SYMBOL (If applicable)<br>NM | 9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER<br>AFOSR-82-0167 |
| 8c. ADDRESS (City, State and ZIP Code)<br>Bolling AFB DC 20332 | 10. SOURCE OF FUNDING NOS.<br>PROGRAM ELEMENT NO.: 61102F<br>PROJECT NO.: 2304<br>TASK NO.: A5<br>WORK UNIT NO.: |
| 11. TITLE (Include Security Classification)<br>AN OPERATOR METHOD FOR COMPUTING THE ASYMPTOTICS OF A COLLISION RESOLUTION INTERVAL | |
| 12. PERSONAL AUTHOR(S)<br>Walter A. Rosenkrantz | |
| 13a. TYPE OF REPORT<br>Technical<br>13b. TIME COVERED<br>FROM ______ TO ______ | 14. DATE OF REPORT (Yr., Mo., Day)<br>29 AUG 84<br>15. PAGE COUNT<br>18 |
| 16. SUPPLEMENTARY NOTATION | |
| 17. COSATI CODES<br>FIELD / GROUP / SUB. GR. | 18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)<br>Collision resolution algorithms; expected packet delay and expected length of collision resolution interval. |

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The investigators present an operator method for obtaining upper and lower bounds for the expected length of a collision resolution interval for various protocols. The method is elementary in that it circumvents the intricate and ingenious complex variable methods of Fayolle, Flajolet and Hofri (1983). It is also noted that the method can be applied to computing upper and lower bounds for the conditional delay. The problem of computing upper and lower bounds for the variances by this method remains open.

| | |
|---|---|
| 20. DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐ | 21. ABSTRACT SECURITY CLASSIFICATION<br>UNCLASSIFIED |
| 22a. NAME OF RESPONSIBLE INDIVIDUAL<br>MAJ Brian W. Woodruff | 22b. TELEPHONE NUMBER (Include Area Code)<br>(202) 767-5027<br>22c. OFFICE SYMBOL<br>NM |

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

84 10 23 198

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

## 1. Introduction

Ever since the publication of the collision resolution algorithms (CRA) of Capetanakis-Tsybakov-Mikhailov (CTM) there has been a growing interest in the performance analysis of these channel access algorithms (also called protocols). The particular class of algorithms with which this paper is concerned has been clearly described in [FFH 1983] which we recall here.

(1.1) A single broadcast channel is shared among an infinite number of independent sources which emit packets. Channel time is discrete and it takes one interval of time to transmit one packet. Sources are synchronized so that transmissions are initiated at the beginning of each time slot $t = 0,1,2,\ldots$ . We denote by $X$ the number of new packets generated in a time slot and assume that $X$ has a Poisson distribution with parameter $\lambda$ so $E(X) = \lambda$ and variance $V(X) = \lambda$. $X$ is also independent of the past history of the channel.

(1.2) Each transmission is broadcast to every user - including the emitter. When several stations transmit simultaneously, packets will collide and none of them is received correctly. When $n$ packets collide this is called "a collision of multiplicity $n$". Each user involved in the collision uses the same algorithm for retransmitting its blocked packet and they must resolve the collision without the benefit of any other source of information on other users except for information concerning the channel itself, which provides ternary feedback: idle, successful transmission, collision.

The first algorithm proposed for resolving the collision is the ALOHA algorithm which is now known to be unstable and we refer the reader to the author's paper [Ro-To 1983] for a proof using elementary martingale theory. The first class of stable algorithms were developed independently by [Ca 1979], [T-M 1978] with additional interesting variations due to Massey [MA 1981], and Tsybakov-Vvedenskaya [T-V 1980]. Of primary importance in each of these papers is the determination of $\lambda_{max}$ = maximum allowable arrival rate of new packets. In particular for $\lambda < \lambda_{max}$ the system is stable - equivalently, the Markov chain which describes the system is positively recurrent. As noted by several authors including [Ma 1981], [FFH 1983] the only quantity one needs in order to evaluate $\lambda_{max}$ is the expected length of the collision resolution interval (CRI) determined by the CRA. More precisely let $\ell(n)$ denote the random length of time required by the CRA to resolve a collision of multiplicity $n$. The CTM algorithm, see [Ma 1981], leads to the recurrence relation

$$(1.3)\qquad \begin{cases} \ell(n) = 1 + \ell(S_n) + \ell(n - S_n), & n \geq 2 \\ \ell(0) = \ell(1) \end{cases}$$

where $S_n$ is a binomially distributed random variable with $P(S_n = j) = \binom{n}{j} 2^{-n}$

Setting $L(n) = E(\ell(n))$ and using the fact that

$$\begin{aligned} E(\ell(n) &= \sum_{i=0}^{n} E(\ell(n) | S_n = i) \binom{n}{i} 2^{-n} \\ &= \sum_{i=0}^{n} (1 + L(i) + L(n - 1)) \binom{n}{i} 2^{-n} \\ &= 1 + 2 \sum_{i=0}^{n} L(i) \binom{n}{i} 2^{-n} . \end{aligned}$$

Consequently $L(n)$ satisfies the recurrence relation

$$(1.4)\qquad \begin{cases} L(n) = 1 + 2 \sum_{i=0}^{n} L(i) \binom{n}{i} 2^{-n}, & n \geq 2 \\ L(0) = L(1) = 1 & . \end{cases}$$

Remark: This is equation (3.12) on p. 85 of [Ma 1981].

Similarly the CTM algorithm with continuous entry satisfies a recurrence relation of the form:

$$(1.5)\qquad \begin{cases} \ell(n) = 1 + \ell(S_n + X) + \ell(n - S_n + Y) \\ \ell(0) = \ell(1) = 1 \end{cases}$$

where $S_n$, $X$, $Y$ are mutually independent;

$$P(X = j) = P(Y = j) = \exp(-\lambda)\lambda^j/j!, \quad j = 0,1,\ldots$$

and $S_n$ is binomially distributed as before with parameters $n$, $p = 1/2$. Proceeding in exactly the same way as we did in deriving (1.4) we see that $L(n) = E(\ell(n))$ satisfies the system of equations (with infinitely many unknowns!)

$$(1.6)\qquad \begin{cases} L(n) = 1 + 2 \sum_{j=0}^{\infty} L(j) \cdot p_{nj}, & n \geq 2 \\ L(0) = L(1) = 1 \end{cases}$$

where

$$(1.7)\qquad p_{nj} = P(S_n + X = j) = \sum_{i=0}^{j} \binom{n}{i} 2^{-n} \exp(-\lambda)\lambda^{j-i}/(j-i)!; \quad j \geq 0.$$

For future reference we note that

(1.8) $$p_{n0} = 2^{-n}e^{-\lambda} \quad \text{and} \quad p_{n1} = 2^{-n}e^{-\lambda}(\lambda + n).$$

It is worth pointing out that similar but more complicated recurrence relations for the expected delay have been derived by [FFHJ 1983] and it would therefore seem to be quite useful to present an elementary method for studying the existence, uniqueness and asymptotic properties of solutions to equations of the form (1.6). It is the purpose of this paper to present just such a method, a percursor of which may already be found in a previous paper of the author [Ro 1973]. The method is elementary in the sense that the intricate and ingenious complex variables methods of [FFH 1983] are circumvented. It is to be observed that a similar idea also appears in [T-V 1980] where they refer to the notion of a "barrier function".

The paper is organized as follows: In part 2 we prove an existence and uniqueness theorem for equation 1.6. In particular for $\lambda < \lambda_{max} = (-5 + \sqrt{41})/4 = .35078$ we show that there exists a unique nonnegative solution $L(n)$ to (1.6) satisfying the growth condition

(1.9) $$L(n) \le a \cdot n + b \quad \text{for} \quad n \ge 2.$$

Some additional results obtained include:

(i) a lower bound for $L(n)$ of the form

(1.10) $$L(n) \ge [2(n - \lambda) - 1]/(1 - 2\lambda), \quad n \ge 2, \quad \lambda < \lambda_{max}.$$

For example when $\lambda = .34$ we obtain

(1.11) $$L(n) \ge 6.25n - 5.25, \quad n \ge 2.$$

On the other hand when $L(n)$ is the solution to (1.4) then it can be shown, cf. [Ma 1981, p. 87] that

$$L(n) \leq 3n \quad \text{for} \quad n \geq 2. \tag{1.12}$$

This confirms in a quantitative way one's intuition that the length of a CRI is longer on the average for the continuous entry scheme than it is for the CRA with blocked access. The lower bounds (1.10) and (1.11) appear to be new.

(ii) A short and simple proof of the intuitively obvious fact than $L(n)$ is monotone increasing in $n$ - see lemma 2.19.

(iii) Simplifications and improvements in the statements and proofs of Theorem 2 of [T-V 1980]. In particular their condition (4.5) is shown to be unnecessary for the validity of their theorems.

In part 3 we turn our attention to the sequence $L(n)$ defined by the recurrence relation (1.4) already treated in some detail in [Ma 1981]. In that paper Massey conjectured (see inequality 3.31 of [Ma 1981]) that

$$L(i) + L(n - i) \leq L(n) + 1, \quad 0 \leq i \leq n \tag{1.13}$$

and used this inequality to derive upper and lower bounds for the variance $V(n)$. It seems not to have been noticed that the validity of (1.13) implies the existence of $\lim_{n\to\infty} L(n)/n$ which conflicts with assertions in the literature that this limit does not exist. Thus Massey's derivations of upper and lower bounds for $V(n)$ seem, to this atuhor at least, to be incomplete.

## 2. Existence and Uniqueness of Solutions to Equation (1.6)

To simplify the tyepsetting we denote a sequence by $f(0), f(1), \ldots$ instead of the more customary notation $f_0, f_1, \ldots$ and define the positive linear operator $G_n$ acting on sequences

$f(n)$, $n = 0,1,2$ via the recipe

$$G_n f \triangleq 2 \sum_{j=0}^{\infty} f(j) p_{nj} = 2E(f(S_n + X)). \tag{2.1}$$

In this notation the infinite system of equations (1.6) can be re-written in the more convenient operator form

$$\begin{cases} L(n) = 1 + G_n L, & n \geq 2 \\ L(0) = L(1) = 1 & . \end{cases} \tag{2.2}$$

We can get rid of the constant 1 appearing on the right hand side of (2.2) via the change of variables $f(n) = 1 + L(n)$, $n \geq 0$ and it is easy to see that $f(n)$ satisfies the system

$$\begin{cases} f(n) = G_n f \\ f(0) = f(1) = 2 & . \end{cases} \tag{2.3}$$

We begin by noting that the function $h(n) = M(n - 2\lambda)$, $n = 0,1,2,\ldots$ satisfies the equation (2.3) but with slightly different initial conditions: $h(0) = -2\lambda M$, $h(1) = M(1 - 2\lambda)$. To see this just compute

$$G_n h = 2E(h(S_n + X)) = 2ME(S_n + X) - 4M\lambda$$

$$= 2M(n/2) + \lambda, - 4M\lambda = M(n - 2\lambda) = h(n).$$

Note: Throughout this paper we assume $0 < \lambda < 1/2$. Notice that $h(n)$ is linear in $n$ and it is reasonable to conjecture that $f(n)$ itself is nearly linear in the sense that $f(n) \leq a \cdot n + b$ as $n \to \infty$. Unlike the system of equations (1.4) for which uniqueness and existence is trivial it is not at all a priori evident that solutions to (2.3) are unique. We begin therefore with a proof of uniqueness since this requires the introduction of a technique that will serve us well in our existence proof.

Theorem: There is at most one nonnegative solution $f(n)$ to (2.3) satisfying the growth condition $|f(n)| \leq a \cdot n + b$.

Proof: Suppose $f_1(n)$ and $f_2(n)$ are two solutions to (2.3) then $W(n) = f_1(n) - f_2(n)$ satisfies

$$(2.4) \qquad \begin{cases} w(n) = G_n w, & n \geq 2 \\ w(0) = w(1) = 0 \text{ and } |w(n)| \leq a'n + b' \end{cases}$$

Thus it suffices to show that the only solution to (2.4) is $w(n) \equiv 0$. This follows at once from the existence of a "barrier function" $\rho(n) = h(n) + n^2$, $n \geq 2$, $\rho(0) = \rho(1) = 0$ with the following properties:

$$(2.5) \qquad \begin{cases} \text{(i)} & \rho(n) > 0, \quad n \geq 2 \\ \text{(ii)} & \lim\limits_{n \to \infty} |w(n)|/\rho(n) = 0 \\ \text{(iii)} & 2 \sum\limits_{j=2}^{\infty} \rho(j) p_{nj} = \cancel{G_n \rho,} \ \rho(n) \quad n \geq 2 \ . \end{cases}$$

(i) is obvious and (ii) follows from the fact that $\lim\limits_{n \to \infty} n^{-2} w(n) = 0$, while $1 - 2\lambda > 0$. Later we shall have to restrict $\lambda$ even further and assume $\lambda < \lambda_{max} = .35078$. We postpone for the moment a proof of (iii) and proceed to derive the uniqueness theorem by exploiting the properties of $\rho$. Indeed the assumption that $w(n) \not\equiv 0$ leads to a contradiction as we now show. Condition (ii) implies that if $w(n)$ is not identically zero then there exists a finite integer $n' \geq 2$ with the property $\sup_{n\geq 2} |w(n)|/\rho(n) = |w(n')|/\rho(n') > 0$. From equation (2.4) we see that

$$|w(n')| \leq 2 \sum_{j=2}^{\infty} [|w(j)|/\rho(j)]\rho(j)p_{n'j}$$

$$\leq (|w(n')|/\rho(n'))G_{n'}\rho < |w(n')|$$

where we have used (2.5 iii) in the last step. This is a contradiction so we conclude $\sup_{n\geq 2} |w(n)|/\rho(n) = 0$ i.e. $w(n) \equiv 0$.

We now prove that $\rho(n)$ satisfies property (2.5 iii). Since $2 \sum_{j=1}^{\infty} j^2 p_{nj}$ is twice the second moment of the random variable $S_n + X$ whose variance is equal to $n/4 + \lambda$ we see at once that $2 \sum_{j=1}^{\infty} j^2 p_{nj} = n/4 + \lambda + (n/2 + \lambda)^2$. Consequently

$$(2.6) \qquad 2 \sum_{j=2}^{\infty} j^2 p_{nj} = 2[(n/4 + \lambda) + (n/2 + \lambda)^2] - 2p_{n1},$$

$$p_{n1} = 2^{-n} e^{-\lambda}(\lambda + n), \quad \text{cf (1.8).}$$

A similar calculation yields

$$(2.7) \qquad 2 \sum_{j=2}^{\infty} h(j)p_{nj} = h(n) - 2p_{n0}M(-2\lambda) - 2p_{n1}M(1 - 2\lambda) .$$

After some routine algebraic manipulations the term $4M\lambda p_{n0} - 2M(1 - 2\lambda)p_{n1}$ simplifies to $M2^{-n+1}\exp(-\lambda)(2\lambda^2 + \lambda(2n + 1) - n)$. A necessary condition then for (2.5 iii) to hold is that $2\lambda^2 + \lambda(2n+1) - n \leq 0$ for all $n \geq 2$. This requires an examination of the largest positive root $\lambda_{max}(n)$ of the quadratic equation $2\lambda^2 + \lambda(2n+1) - n$ which turns out to be

$$\lambda_{max}(n) = [-(2n+1) + \sqrt{4n^2 + 12n + 1}\,]/4 \,. \tag{2.8}$$

Notice that $4n^2 + 12n + 1 > (2n+2)^2$ for $n \geq 2$ and therefore $\lambda_{max}(n) \geq .25$. Actually we can do a lot better since it is easy to see that the smallest value of $\lambda_{max}(n)$ occurs at $n = 2$ with $\lambda_{max}(2) = .35078 = \lambda_{max}$. Summing up then we have shown that for $\lambda < .35078$

$$2\sum_{j=2}^{\infty} h(j)p_{nj} - h(n) \leq M2^{-n+1}\exp(-\lambda)(2\lambda^2 + \lambda(2n+1) - n) < 0. \tag{2.9a}$$

The next step is to study $\left(2\sum_{j=2}^{\infty} j^2 p_{nj} - n^2\right)$ for $n \geq 2$. From (2.6) we see that

$$\begin{aligned} 2\sum_{j=2}^{\infty} j^2 p_{nj} - n^2 &= (n/2 + 2\lambda + n^2/2 + 2n\lambda + 2\lambda^2 - n^2 \\ &= n/2 + 2\lambda + 2n\lambda + 2\lambda^2 - n^2/2. \end{aligned} \tag{2.9b}$$

A simple calculation shows that the right hand side of (2.10) is strictly negative for all $n \geq 4$ and $0 \leq \lambda \leq .5$. And it is also clear that for any fixed $\lambda < \lambda_0$ and $n = 2,3$ we can choose $M$ (depending on $\lambda$) sufficiently large and positive so that $2\sum_{j=2}^{\infty}(h(j) + j^2)p_{nj} < h(n) + n^2, n \geq 2$ and this completes the proof of uniqueness.

Remark: The function $\rho(n) = h(n) + n^2$ for $n \geq 2$ with $\rho(0) = \rho(1) = 0$ satisfies the inequality $G_n\rho < \rho(n)$ and is called a "barrier function" by Tsybakov-Vvedenskaya. By constructing barrier function $\rho(n)$ tending to infinity at the rate $n^2$ the uniqueness proof of Tsybakov-Vvedenskaya is considerably simplified as the reader may easily check for himself. Similar ideas were used (but with respect to a different operator $\hat{G}$!) in our paper [Ro 1973].

We now prove the existence of a nonnegative solution to the system of equations

$$(2.10)\qquad \begin{cases} G_n g = g(n) \\ g(0) = a_0,\ g(1) = a_1,\ a_i \geq 0,\ i = 0,1. \end{cases}$$

Remark: The system (2.3) corresponds to the special case $a_0 = a_1 = 2$. Our proof is similar to that in [T-V 1980] except that a judicious use of the Lebesgue dominated convergence theorem yields a more *streamlined* proof. We need the following

(2.11) Lemma: Let $X^o(n) = h(n)$, $n \geq 2$, and $X^o(0) = a_0$, $X^o(1) = a_1$. Then for $\lambda < \lambda_0$ $X^o(n)$ is a barrier function i.e. $G_n X^o \leq X^o(n)$, $n \geq 2$, provided $M = M(\lambda, a_0, a_1)$ defined at (2.15) below.

Proof: The necessary calculations are similar to those already carried out at (2.7). Since $X^o(j) - h(j) \neq 0$ only for $j = 0,1$ we see that

$$G_n(X^o - h) = (X^o(0) - h(0))p_{n0} + (X^o(1) - h(1))p_{n1} = r(n,\lambda).$$

Thus for $n \geq 2$ we have

$$(2.12)\qquad G_n X^o = G_n h + G_n(X^o - h) = X^o(n) + r(n,\lambda),\quad n \geq 2.$$

Consequently $r(n,\lambda) \le 0$, $n \ge 2$ is a necessary and sufficient condition for $X^o$ to be a barrier function. In particular

$$(2.13)\qquad r(n,\lambda) = M2^{-n}e^{-\lambda}(2\lambda^2 + \lambda(2n+1) - n) + a_0\, e^{-\lambda}2^{-n} + a_1 e^{-\lambda}\lambda 2^{-n}(\lambda + n)$$
$$= 2^{-n}e^{-\lambda}\{M(2\lambda^2 + \lambda(2n+1) - n) + a_0\lambda + a_1\lambda(\lambda + n\} .$$

Since $a_0$ and $a_1$ are nonnegative it is clear that a necessary condition for $r(n,\lambda) \le 0$ all $n \ge 2$ is that $2\lambda^2 + \lambda(2n+1) - n < 0$ - equivalently for $\lambda < \lambda_0 = \lambda_{max}(2) = .35078$ (cf (2.8)). Keeping in mind that $n - (2\lambda^2 + \lambda(2n+1)) > 0$ for $\lambda < \lambda_0$ we choose

$$(2.14)\qquad M > (a_1\lambda n + a_0 \;+ a_1\lambda^2)/(n - (2\lambda^2 + \lambda(2n+1))), \quad n \ge 2,$$

and it is easy to see that the right hand side of (2.14) remains bounded as $n \to \infty$. So it suffices to choose

$$(2.15)\qquad M(\lambda,a_0,a_1) = \sup_{n\ge 2} [(a_1\lambda n + a_0\lambda + a_1\lambda^2)/(n - (2\lambda^2 + \lambda(2n+1)))].$$

This completes the proof of the existence of a barrier function.

Remark: Notice that $G_n X^o \le X^o(n)$ implies that $X^o$ is a nonnegative function integrable in the sense of Lebesgue with respect to the "weights" $\mu_n(j) = 2p_{nj}$ since $G_n X^o = \sum_{j=0}^{\infty} X^o(j)\mu_n(j) \le X^o(n)$. We shall make use of this remark shortly.

Equipped with the barrier function $X^o(n)$ of lemma (2.11) it is fairly straightforward to construct a solution to equation (2.10). Define the sequence of functions $X^k(n)$ via the recipe

$$X^k(n) = G_n X^{k-1}, \quad n \geq 2 \tag{2.16}$$

$$X^k(0) = a_0, \quad X^k(1) = a_1.$$

(2.17) Assertion: $0 \leq X^{k+1}(n) \leq X^k(n)$, $k = 0,1,2,\ldots$ and therefore $\lim_{k\to\infty} X^k(n) = g(n)$ exists. Moreover $G_n g = g(n)$ for $n \geq 2$ and $g(n) \leq X^o(n) = M(\lambda, a_0, a_1)(n - 2\lambda)$, so $|g(n)| \leq a \cdot n + b$.

Proof: $X^1(n) = G_n X^o \leq X^o(n)$ implies $X^2(n) = G_n X^1 \leq G_n X^o = X^1(n) \leq X^o(n)$ and so on. Thus $X^k(n)$ is monotone decreasing in $k$ as claimed and since the sequence is bounded below it follows that $g(n) = \lim_{k\to\infty} X^k(n)$ exists and satisfies the inequality $|g(n)| \leq a \cdot n + b$. The only thing we have to check is that $G_n g = g(n)$ for $n \geq 2$. Observe that $0 \leq X^k(j) \leq X^o(j)$ and $\sum_{j=0}^{\infty} X^o(j) 2p_{nj} \leq X^o(n)$ permit us to use the Lebesgue dominated convergence theorem to conclude

$$\lim_{k\to\infty} \sum_{j=0}^{\infty} X^k(j) 2p_{nj} = \sum_{j=0}^{\infty} (\lim_{k\to\infty} X^k(j)) 2p_{nj} \tag{2.18}$$

i.e. $\lim_{k\to\infty} G_n X^k = G_n(\lim X^k) = G_n g$. But $|g(n) - G_n g| \leq |g(n) - X^{k+1}(n)| + |X^{k+1}(n) - G_n g|$.

Clearly $\lim_{k\to\infty} |g(n) - X^{k+1}(n)| = 0$ and $\lim_{k\to\infty} |X^{k+1}(n) - G_n g| = \lim_{k\to\infty} |G_n X^k - G_n g| = 0$ using (2.18) for the last step. This completes the proof of the existence of a solution to (2.10) satisfying the growth condition $|g(n)| \leq a \cdot n + b$, consequently the solution is unique.

We now proceed to use the existence and uniqueness theorems to derive the lower bound (1.10).

Set $w(n) = f(n) - h(n)$ where $f$ is the unique solution to (2.3) satisfying the growth condition $|f(n)| \leq an + b$ and $h(n) = M(n - 2\lambda)$. It is easy to check that for $0 < M < 2(1 - 2\lambda)^{-1}$ we have $w(0) > 0$, $w(1) > 0$ with $G_n w = w(n)$, $n \geq 2$ and $|w(n)| \leq an + b$. Consequently by the uniqueness theorem $w(n) \geq 0$ all $n \geq 2$ i.e.

$$f(n) \geq h(n) = [2(1 - 2\lambda)^{-1} - \varepsilon](n - 2\lambda) \quad \text{for} \quad n \geq 2.$$

Since this is true for every $\varepsilon > 0$ we get $f(n) \geq (2(1 - 2\lambda)^{-1})(n - 2\lambda)$. The lower bound for $L(n)$ is now derived by noting that $L(n) = f(n) - 1$.

Finally we show how the probabilistic representation $G_n g = 2E(g(S_n + X))$ may be used to show that the solutions to (2.10) are monotone increasing in $n$.

(2.19) Lemma: If $g(n)$ is monotone increasing and nonnegative then so is $G_n g$.

Proof: $G_{n+1}g - G_n g = 2E\{g(S_{n+1} + X) - g(S_n + X)\}$, so it suffices to show that $E(g(S_{n+1} + X) - g(S_n + X)) \geq 0$. But

$$E(g(S_{n+1} + X) - g(S_n + X) | S_n = i, X = j) =$$
$$(1/2\ )(g(i + j + 1) - g(i + j)) \geq 0 \quad \text{for every} \quad i,j.$$

Equivalently

$$(2.20) \qquad E(g(S_{n+1} + X) - g(S_n + X) | F_n) \geq 0$$

where $F_n$ = smallest sigma-field with respect to which both $S_n, X$ are measurable. Taking expectations of both sides of (2.20) yields the desired inequality that $G_{n+1}g - G_n g \geq 0$. In particular then each function $X^k(n)$ is monotone increasing in $n$ since $X^o(n)$ is obviously monotone. Consequently so is the limit $g(n) = \lim_{k\to\infty} X^k(n)$.

3. Remarks on the asymptotic behaviour of the solution to (1.4)

In the postscript to [Ma 1981] reports that Vvedenskaya has shown (so far unpublished) that $\lim_{n\to\infty} L(n)/n$ does not exist although it can be shown (and we shall sketch such a proof below based on the methods in the preceding section of this paper) that

(3.1) $$2.881 \le \liminf_{n\to\infty} L(n)/n \le \limsup_{n\to\infty} L(n)/n \le 2.8966.$$

In the same paper Massey asserted, without proof, that

(3.2) $$L(i) + L(n - i) \le L(n) + 1, \quad 0 \le i \le n.$$

In September of 1983 I pointed out to my colleagues Don Towsley and Jack Wolf of the Department of Electrical and Computer Engineering here at the University of Massachusetts that any inequality of the form:

(3.3) $L(i) + L(n - i) \le L(n) + c, \quad 0 \le i \le n$ where $c$ is an arbitrary constant, necessarily implies the existence of $\lim_{n\to\infty} L(n)/n$.

In other words the nonexistence of the $\lim_{n\to\infty} L(n)/n$ is inconsistent with the validity of (3.3). To see this we need to introduce the notion of a subadditive sequence.

(3.4) Definition: We say that a sequence $a_n$ is subadditive if $a_n + a_m \ge a_{n+m}$ holds.

(3.5) Theorem: If $a_n$ is a subadditive sequence then $\lim_{n\to\infty} a_n/n$ exists.

Proof: See [P-SZ (1972) p.23, problem 98].

Application: Suppose (3.3) holds then the sequence $a_n = c - L(n)$ satisfies the condition $a_m + a_{n-m} \geq a_n$ which is equivalent to the condition $a_m + a_n \geq a_{m+n}$ i.e. $a_n$ is subadditive and therefore $\lim_{n\to\infty} a_n/n = \lim_{n\to\infty} L(n)/n$ exists.

Now inequality (3.2) was used by Massey to derive upper and lower bounds on the asymptotic behaviour of the variance of the CRI length $V(n)$ defined by (1.3). These proofs must now be regarded as incomplete.

Turning now to the proof of (3.1) we shall content ourselves with a sketch of the derivation of the upper bound

$$\limsup_{n\to\infty} L(n)/n \leq 2.8966. \tag{3.6}$$

As we did in part 2 of this paper we set $f(n) = L(n) + 1$ and study the asymptotic behaviour of the recurrence relation

$$\begin{cases} f(n) = 2E(f(S_n)), & n \geq 2 \\ f(0) = f(1) = 2 . \end{cases} \tag{3.7}$$

We shall make extensive use of the following elementary lemma the proof of which is left to the reader.

(3.8) Lemma: Suppose $W(i) \geq 0$, $0 \leq i \leq m$ and $W(n) \geq 2E(W(S_n))$, $n \geq m$. Then $W(n) \geq 0$ all $n$.

Bring in the function $H(n) = \alpha.n$ which satisfies the recurrence relation (3.7) for all $n$. Set $g(n) = H(n)$, $n \geq 4$ and $g(n) = f(n)$, $n \leq 3$ and compute $E(g(S_n) - H(S_n)) = r(n)$. Using the fact that $f(0) = f(1) = 2$, $f(2) = 6$ we see that

$$(3.9) \qquad r(n) = \sum_{j=0}^{3} (g(j) - H(j)) \binom{n}{i} 2^{-n}$$

$$= 2^{-n} n(n - 1)\ 2/(n(n - 1)) + (2 - \alpha)/(n - 1) + (3 - \alpha)$$

$$+ (n - 2)(13/9 - \alpha/2)\ .$$

The problem then is to choose $\alpha$ in such a way that $r(n) \leq 0$ for $n \geq 4$ and $W(n) = g(n) - f(n) \geq 0$ for $0 \leq n \leq 4$. From (3.9) we see that a necessary condition is for $13/9 - \alpha/2 < 0$ or $\alpha > 2.8889$. It turns out $\alpha = 2.8889$ is too small and that we must choose $\alpha = 23/9 + .00778 = 2.8966$. With this choice of $\alpha$ one can show by means of a straightforward but tedious calculation that $W(n) = g(n) - f(n)$ satisfies the conditions of lemma 3.8 and hence $f(n) \leq g(n) = 2.89666n$ for $n \geq 4$. In a similar calculation we can derive Massey's lower bounds (Table 3.2 on P. 87 of [Ma 1981]).

## 4. Concluding remarks.

In this paper we have presented an operator method for obtaining upper and lower bounds for the expected length $L(n)$ of a CRI for various protocols. Although we shall not carry out such an analysis here it is noteworthy that the same methods can be used to obtain upper and lower bounds for the expected delay $W(n)$ = expected delay experienced by a packet that had its first transmission during a collision of multiplicity $n$. As pointed out by [FFHJ (1983)] in the case of CTMCRA with blocked access $w(n)$ (with $W(n) = E(w(n))$) satisfies the recurrence relation

$$(4.1) \qquad \begin{cases} w(n) = w(S_{n-1} + 1) & \text{with probability } 1/2 \\ w(n) = w(n - S_{n-1}) + \ell(S_{n-1}) & \text{with probability } 1/2. \end{cases}$$

Taking expectations as before we are led to a more complicated system of recurrence equations whose asymptotics can be studied by means of similar methods. We intend to pursue these questions in another paper.

We must however point out that our methods have so far been unable to give a proof of Massey's upper and lower bounds for $V(n)$ = Variance $(\ell(n))$. Obtaining upper and lower bounds for the second and higher order moments of $\ell(n)$ is one of the more interesting and still open problems in this area.

References

[CA 1979] J. I. CAPETANAKIS, "Tree algorithms for packet broadcast channels", IEEE Trans. Inf. Theory, Vol. IT - 25 pp. 205-515 (1979).

[FFH 1983] G. FAYOLLE, P. FLAJOLET, M. HOFRI, "On a functional equation arising in the analysis of a protocol for a multi-access broadcast channel", (Preprint).

[FFHJ 1983] G. FAYOLLE, P. FLAJOLET, M. HOFRI, P. JACQUET, "The evaluation of packet transmission characteristics in a multiaccess channel with stack collision resolution protocol", Rapports de Recherche No. 245, INRIA, October (1983).

[MA 1981] J. L. MASSEY, "Collision resolution algorithms and random access communications", in Multi-User Communication Systems, edited by G. Longo, Springer-Verlag, New York (1981).

[P-SZ 1972] G. PÓLYA and G. SZEGÖ, "Problems and theorems in Analysis I, Springer Verlag Berlin (1972).

[Ro 1973] W. ROSENKRANTZ, "A method for computing the asymptotic limit of a class of expected first passage times", Annals of Probability, Vol. 1, No. 6 pp. 1035-1043, (1973).

[Ro-To 1983] W. ROSENKRANTZ, D. TOWSLEY, "On the instability of the slotted ALOHA multiaccess algorithm", IEEE Trans. on Automatic Control, Vol. AC-28, No. 10, pp. 994-996, (1983).

[T-M 1978] B. S. TSYBAKOV, V. A. MIKHAILOV, "Free synchronous packet access in a broadcast channel with feedback", Problems of Information Transmission (English Translation) Vol. 14, No. 4, p. 259-280, (1978).

[T-V 1980] B. S. TSYBAKOV, N. D. VVEDENSKAYA, "Random multiple access algorithms", Problems of Information Transmission (English Translation), Vol. 16, No. 3, pp. 230-243, (1980).

END

FILMED

11-84

DTIC